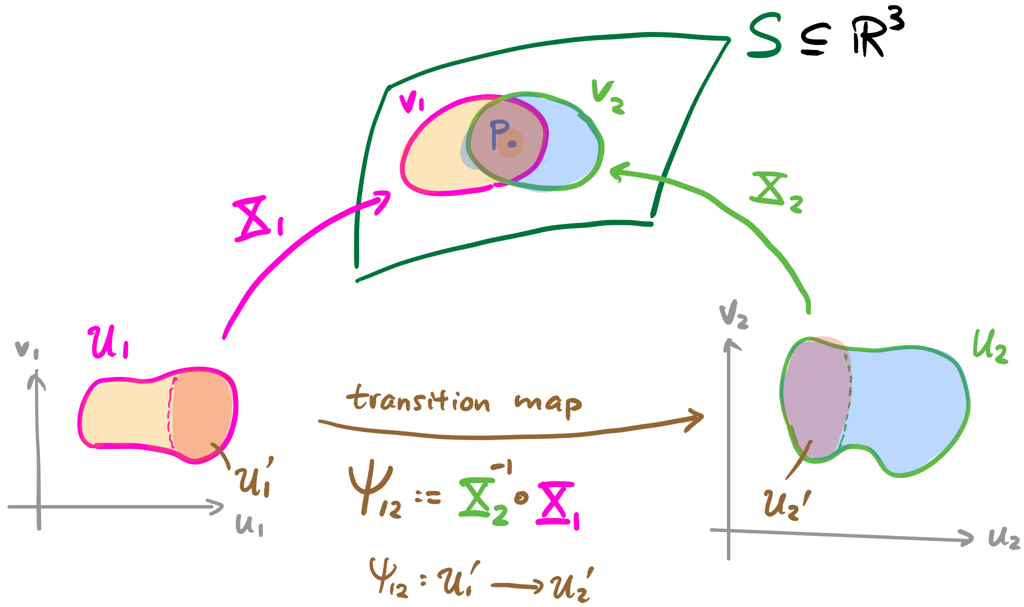
§ Change of parameters / coordinates (do Carmo § 2.3)

Consider a point $p \in S$ covered by two coordinate systems



Claim: $\varphi_{12} : \mathcal{U}_1' \to \mathcal{U}_2'$ is a diffeomorphism (between open sets in $\mathbb{R}^2$)

Proof: Clearly, φ_{12} is a homeomorphism with

inverse $\varphi_{12}^{-1} = \varphi_{21} := \mathbf{X}_1^{-1} \circ \mathbf{X}_2 : \mathcal{U}_2' \to \mathcal{U}_1'$

It remains to show that φ_{12} is smooth.

Since S is locally a graph, say over xy-plane.

Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection map onto xy-plane.

Then,
$$\varphi_{12} = \mathbf{X}_2^{-1} \circ \mathbf{X}_1 = \underbrace{(\pi \circ \mathbf{X}_2)^{-1}}_{\mathbb{R}^2 \to \mathbb{R}^2} \circ \underbrace{\pi}_{\mathbb{R}^3 \to \mathbb{R}^2} \circ \underbrace{\mathbf{X}_1}_{\mathbb{R}^2 \to \mathbb{R}^3} \quad \text{Smooth!}$$

smooth maps on open subsets of

$\square$

§ Differentiability (do Carmo § 2.3)

We now define the concept of "differentiability" of functions into and out of a surface $S \subseteq \mathbb{R}^3$.

<u>Def$\underline{n}$</u>: A function $f : S \longrightarrow \mathbb{R}^n$ is smooth

if $f \circ \mathbf{X} : \mathcal{U} \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^n$ is smooth

for <u>ANY</u> parametrization $\mathbf{X} : \mathcal{U} \subseteq \mathbb{R}^2 \longrightarrow S$

<u>Remark</u>: Since change of coordinates φ_{12} are smooth diffeomorphisms, we just have to check smoothness on <u>SOME</u> collection of parametrizations covering S.

<u>Example</u>: (Restriction of smooth functions on $\mathbb{R}^3$)

Let $f : V \overset{\text{open}}{\subseteq} \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a smooth function

s.t. $S \subseteq V$, then

$$f|_S : S \longrightarrow \mathbb{R} \quad \text{is smooth.}$$

In particular, the restrict of the coordinate functions x, y and z to S are smooth functions.

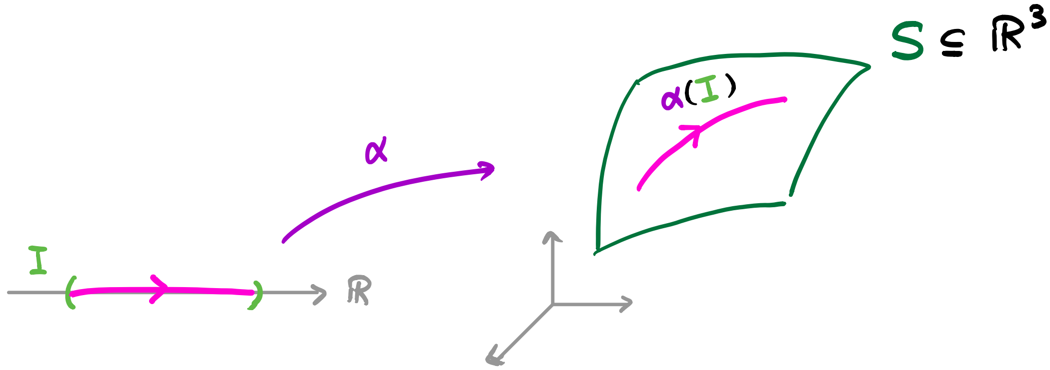
<u>FACT</u>: $\underset{\text{smooth}}{f, g : S \to \mathbb{R}} \implies \underset{\substack{\text{smooth} \\ \text{(if well-defined)}}}{f \pm g,\ fg,\ f/g}$

Def$^{\underline{n}}$: A map $f : \mathcal{U} \subseteq \mathbb{R}^n \longrightarrow S$ is smooth
if it is smooth as a map $f : \mathcal{U} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^3$.

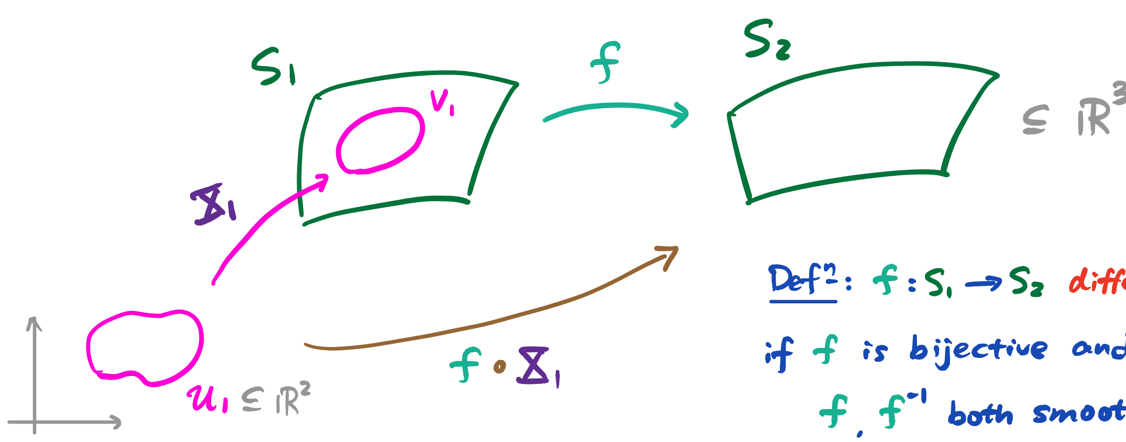
Example: Curves on a surface

Let $\alpha : I \longrightarrow \mathbb{R}^3$ be a space curve with $\alpha(I) \subseteq S$

Then, $\alpha : I \subseteq \mathbb{R} \longrightarrow S$ is smooth.



Def$^{\underline{n}}$: A map $f : S_1 \longrightarrow S_2$ between two surfaces is smooth if $f \circ \mathbf{X}_1 : \mathcal{U}_1 \subseteq \mathbb{R}^2 \longrightarrow S_2 \subseteq \mathbb{R}^3$ is smooth for any parametrization $\mathbf{X}_1$ of S_1.



Def$^{\underline{n}}$: $f : S_1 \to S_2$ diffeomorphism if f is bijective and f, f^{-1} both smooth.